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Bell-Type Inequalities in Orthomodular Lattices. I. Inequalities of Order 2

Anatolij Dvurečenskij[1] and Helmut Länger[2]

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We study Bell-type inequalities of order n with emphasis on the case $n = 2$ in the framework of the structure of an orthomodular lattice, which is a logicoalgebraic model of quantum mechanics. We give necessary and sufficient conditions for the validity of Bell-type inequalities of order 2. In particular, we study Bell-type inequalities in various structures connected with a Hilbert space, and we give a characterization of Boolean algebras via the validity of certain Bell-type inequalities.

1. INTRODUCTION

The probability world of classical mechanics may be described in the framework of Boolean algebras as was done in the Kolmogorov (1933) axiomatic model. In contrast to this, quantum mechanics provides a more general structure than Boolean algebras. Bell (1964) gave an example of an inequality involving three probabilities

$$p(a) + p(b) - p(a \wedge b) \leq 1$$

which is valid in classical probability theory but violated by some quantum mechanical experiments. This observation started an intensive investigation of so-called Bell-type inequalities (Clauser *et al.*, 1969; Santos, 1986, 1988; Beltrametti and Maczyński, 1991, 1992a,b, 1994; Pulmannová and Majerník, 1992; Pulmannová, 1994; Länger and Maczyński, n.d.).

Today we use a logicoalgebraic approach to quantum mechanics (Birkhoff and van Neumann, 1936; Varadarajan, 1968), where two essential postu-

[1]Mathematical Institute, Slovak Academy of Sciences, SK-814 73 Bratislava, Slovakia. E-mail: dvurecen@mau.savba.sk.
[2]Technische Universität Wien, Institut für Algebra und Diskrete Mathematik, A-1040 Vienna, Austria. E-mail: hlaenger@email.tuwien.ac.at.

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lates are proposed: (i) to any physical system $\mathcal{F}$ an orthomodular lattice L (also called a quantum logic) is associated, and (ii) any preparation procedure of the physical system defines a state.

By a Bell-type inequality of order n we understand any inequality of the type

$$\sum_{I \subseteq N} f(I) p\left(\bigwedge_{i \in I} a_i\right) \in [0, 1]$$

which holds for a state p in an orthomodular lattice L (and for all $a_1, \ldots, a_n \in L$), where $f(I)$ is a real coefficient, $p(\wedge_{i \in I} a_i)$ is a correlation, or a joint distribution, of the set of events $\{a_i: i \in I\}$ in the state p, $a_1, \ldots, a_n \in L$, and $N := \{1, \ldots, n\}$.

A general approach using orthomodular lattices has been studied in Beltrametti and Maczyński (1994) and Länger and Maczyński (n.d.), where it was shown that there exists an intimate connection between Bell-type inequalities of order n holding in any classical model on the one hand and inequalities of type

$$\sum_{i \subseteq K} f(I) \in [0, 1] \qquad \text{for any} \quad K \subseteq N$$

on the other hand.

This and a subsequent paper are devoted to the investigation of Bell-type inequalities within the framework of orthomodular lattices. In the first paper we study Bell-type inequalities of order 2. It turns out that the validity of as much as possible of such Bell-type inequalities is equivalent to the subadditivity of the corresponding state.

The paper is organized as follows: The basic definitions and notions are given in Section 2, and the definitions of a correlation function and of a Bell-type inequality of order n follow in Section 3. Section 4 contains general criteria for validity of Bell-type inequalities of order 2. In Sections 5 and 6, Bell-type inequalities in various structures connected with Hilbert spaces and in some other orthomodular lattices are studied. The connection between Bell-type inequalities of order 2 and the distributivity of the corresponding lattices is exhibited in Section 7.

We postpone a detailed study of Bell-type inequalities of order at least 3 to the subsequent paper.

2. BASIC DEFINITIONS AND NOTIONS

We shall assume that the event structure of a quantum mechanical measurement is described by a *quantum logic*, or, equivalently, by an *ortho-*

modular lattice (OML) L. So, let L be an OML, i.e., L is a lattice with respect to a partial ordering $\leq$ and with the greatest and least elements 1 and 0 $(0 \neq 1)$, equipped with an *orthocomplementation* $^\perp$: $L \to L$, $a \mapsto a^\perp$, $a, a^\perp \in L$, such that, for all $a, b \in L$: (i) $a^{\perp\perp} = a$; (ii) $a \vee a^\perp = 1$; (iii) if $a \leq b$, then $b^\perp \leq a^\perp$; (iv) if $a \leq b$, then $b = a \vee (b \wedge a^\perp)$ (*orthomodular law*). If an OML L is as a lattice σ-complete or complete, then we say that L is a σ-OML or a complete OML, respectively. For more details on OMLs see Beran (1984), Kalmbach (1983), Pták and Pulmannová (1994), and Dorninger and Müller (1984) and on lattices Birkhoff (1967).

We say that two elements a and b of L are: (i) *orthogonal*, and write $a \perp b$, iff $a \leq b^\perp$; (ii) 0-*orthogonal* if $a \wedge b = 0$; (iii) *compatible*, and write $a \leftrightarrow b$, iff there are three mutually orthogonal elements $a_1, b_1, c \in L$ such that $a = a_1 \vee c$, $b = b_1 \vee c$. Then $a_1 = a \wedge b^\perp$, $b_1 = b \wedge a^\perp$, $c = a \wedge b$. It is possible to show that $a \leftrightarrow b$ iff $a = a \wedge b \vee a \wedge b^\perp$ (Kalmbach, 1983).[3]

A subset L_0 of L containing 0 and 1 is said to be a *sub-OML* of the OML L if $a \in L_0$ implies $a^\perp \in L_0$, and if $a, b \in L_0$ implies $a \vee b \in L_0$, where the joins taken in L_0 and L are the same. If M is a subset of L, then there exists a smallest sub-OML of L, denoted by $L_0(M)$, containing M; indeed, $L_0(M) = \bigcap\{L_0 \subseteq L: L_0 \supseteq M, L_0$ is a sub-OML of $L\}$; it is called the sub-OML of L generated by M. If a sub-OML L_0 of L is distributive, i.e., if for all $a, b, c \in L_0$ we have

$$(a \vee b) \wedge c = a \wedge c \vee b \wedge c \tag{2.1}$$

or

$$(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c) \tag{2.2}$$

then we call it a *Boolean subalgebra* of L.

The center of L is the set $C(L) = \{a \in L: a \leftrightarrow b$ for all $b \in L\}$. It is clear that $0, 1 \in C(L)$, and if $a \in C(L)$, then $a^\perp \in C(L)$. In addition, $C(L)$ is a Boolean subalgebra of L. Moreover, an OML L is a Boolean algebra iff $L = C(L)$. An OML L is said to be *irreducible* iff $C(L) = \{0, 1\}$.

If the event structure L of a physical system $\mathcal{F}$ is a Boolean algebra, we say that $\mathcal{F}$ is a classical system, and if L is not a Boolean algebra, we say that $\mathcal{F}$ is not a classical system. If the event structure of a subsystem $\mathcal{F}_0$ of the physical system $\mathcal{F}$ forms a Boolean subalgebra L_0 of L, we say that $\mathcal{F}_0$ is locally classical in $\mathcal{F}$.

For any pair $a, b \in L$, we define the *commutator*, com(a, b), of a, b via

$$\mathrm{com}(a, b) = a \wedge b \vee a \wedge b^\perp \vee a^\perp \wedge b \vee a^\perp \wedge b^\perp \tag{2.3}$$

Then $a \leftrightarrow b$ iff com$(a, b) = 1$. It is clear that if $a \perp b$, then $a \wedge b = 0$; the converse implication holds in any Boolean algebra. We recall that by

[3] We note that $\wedge$ has a higher priority than $\vee$.

Varadarajan (1968), $L_0(M)$ is a Boolean subalgebra of L iff $a \leftrightarrow b$ for all a, $b \in M$.

One of the most important models of quantum logic theory is the system $L(H)$ of all closed subspaces of a real or complex Hilbert space H (not necessarily separable), where the partial ordering $\leq$ is the set-theoretic inclusion, with the orthocomplementation $^\perp$: $M \mapsto M^\perp := \{x \in H: (x, y) = 0$ for all $y \in M\}$, and with the null subspace $\{0\}$ and the whole space H as the least and greatest elements. $L(H)$ then forms a complete OML.

A mapping $p: L \to [0, \infty)$ is said to be:

- (i) *subadditive* if $p(a \vee b) \leq p(a) + p(b)$, for all $a, b \in L$;
- (ii) a *valuation* if $p(a \vee b) + p(a \wedge b) = p(a) + p(b)$, for all $a, b \in L$;
- (iii) 0-*additive* if $p(a \vee b) = p(a) + p(b)$ whenever $a \wedge b = 0$;
- (iv) *additive* if $p(a \vee b) = p(a) + p(b)$ whenever $a \perp b$;
- (v) σ-*additive* if, for $\{a_i\}_{i=1}^{\infty}$ with $a_i \perp a_j$, $i \neq j$, such that $\vee_{i=1}^{\infty} a_i \in L$, we have $p(\vee_{i=1}^{\infty} a_i) = \Sigma_{i=1}^{\infty} p(a_i)$;
- (vi) *completely additive* if, for any index set I and any system $\{a_i\}_{i\in I}$ with $a_i \perp a_j$ for $i \neq j$, $i, j \in I$, and $\vee_{i\in I} a_i \in L$, we have[4] $p(\vee_{i\in I} a_i) = \Sigma_{i\in I} p(a_i)$;
- (vii) a *state* if p is additive and $p(1) = 1$;
- (viii) *distributive* if $p((a \vee b) \wedge c) = p(a \wedge c \vee b \wedge c)$ for all $a, b, c \in L$;
- (ix) *modular* if $p((a \vee b) \wedge c)) = p(a \vee (b \wedge c))$ for all $a, b, c \in L$ with $a \leq c$;
- (x) *positive* if $p(a) > 0$ whenever $a \in L\backslash\{0\}$;
- (xi) *Jauch–Piron* if $p(a \vee b) = 0$ whenever $p(a) = p(b) = 0$;
- (xii) 0–1-*valued* if $p(L) = \{0, 1\}$.

We denote by $S(L)$ the set of all states on L. We say that a system $\mathcal{P}$ of states on an OML L is: (1) *separating* if $p(a) = 0$ for all $p \in \mathcal{P}$ implies $a = 0$; (ii) *ordering* if $p(a) \leq p(b)$ for all $p \in \mathcal{P}$ implies $a \leq b$; (iii) *unital* if given $a \neq 0$, there exists a state $p \in \mathcal{P}$ such that $p(a) = 1$; (iv) *full* if $p(a) = p(b)$ for all $p \in \mathcal{P}$ implies $a = b$. We note that any unital system is separating.

If L is a Boolean algebra, then any state is 0-additive, subadditive, and a valuation. If L is not a Boolean algebra, there may exist states which are not subadditive. Since subadditive states will play an important role in a discussion of Bell-type inequalities, (see Sections 4 and 5), we give also a deeper analysis of this notion.

[4] $\Sigma_{i\in I} p(a_i) := \sup\{\Sigma_{i\in J} p(a_i)$: J is a finite subset of $I\}$.

Example 2.1. Let L be given by Fig. 1 (this OML is denoted by MO2 and called the Chinese lantern). Every state p on L is of the following form: $p(0) = 0$, $p(a) = \alpha$, $p(a^\perp) = 1 - \alpha$, $p(b) = \beta$, $p(b^\perp) = 1 - \beta$, and $p(1) = 1$, where α, $\beta \in [0, 1]$. Such a p is subadditive iff $\alpha = \beta = 1/2$ (see Proposition 6.1 or 6.6).

Example 2.2. Let $L = L(H)$, $\dim H \geq 2$, and put

$$p(M) := \lambda\|P_M x_1\|^2 + (1 - \lambda)\|P_M x_2\|^2, \qquad M \in L(H)$$

where $1/2 < \lambda \leq 1$, x_1 and x_2 are two orthonormal vectors in H, and P_M is the orthogonal projection from H onto M. Put $x_1' := \sqrt{2}/2(x_1 + x_2)$ and $x_2' := x_2$. Then x_1' and x_2' are linearly independent vectors and they generate a two-dimensional subspace H_0 of H. If M_y denotes the one-dimensional subspace of H generated by a nonzero vector $y \in H$, we have

$$1 = p(H_0) > p(M_{x_1'}) + p(M_{x_2'}) = (3 - 2\lambda)/2$$

so that p is a state which is not subadditive.

Example 2.3. If $L = L(H)$, $1 \leq \dim H = n < \infty$, then p: $L(H) \to [0, 1]$, defined by

$$p(M) = \dim M/n, \qquad M \in L(H) \tag{2.4}$$

is a subadditive state on $L(H)$.

3. CORRELATION FUNCTIONS AND BELL-TYPE INEQUALITIES

According to Beltrametti and Maczyński (1994) and Länger and Maczyński (n.d.), we introduce a correlation function which will be a crucial notion in our considerations of Bell-type inequalities.

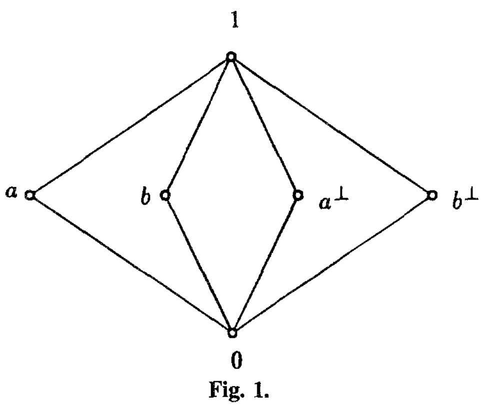

Fig. 1.

Let $N := \{1, \ldots, n\}$ for a fixed integer $n \geq 1$. We put $\wedge_{i \in \varnothing} a_i := 1$ in any OML L.

Proposition 3.1. Let f: $2^N \to \mathbb{R}$ and let p be a state on an OML L. Then the property

$$\sum_{I \subseteq N} f(I) p\left(\bigwedge_{i \in I} a_i\right) \in [0, 1] \qquad \text{for any} \quad a_1, \ldots, a_n \in L \tag{3.1}$$

implies

$$\sum_{I \subseteq K} f(I) \in [0, 1] \qquad \text{for any} \quad K \subseteq N \tag{3.2}$$

Proof. Let $K \subseteq N$ be given. Put $a_i := 1$ if $i \in K$ and $a_i := 0$ if $i \in N \backslash K$. Then

$$[0, 1] \ni \sum_{I \subseteq N} f(I) p\left(\bigwedge_{i \in I} a_i\right) = \sum_{I \subseteq K} f(I) p(1) = \sum_{I \subseteq K} f(I) \quad \blacksquare$$

Proposition 3.2. Let f: $2^N \to \mathbb{R}$ and let p be a state on an OML L. Then the statement

$$\sum_{I \subseteq N} f(I) p\left(\bigwedge_{i \in I} a_i\right) \in [0, 1] \qquad \text{for any} \quad a_1, \ldots, a_n \in \{0, 1\}$$

is equivalent to (3.2).

Proof. This follows ideas of the proof of Proposition 3.1. ■

Corollary 3.3. Propositions 3.1 and 3.2 also hold if f: $2^N \to \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers.

Proof. This is a particular case of Propositions 3.1 and 3.2. ■

Beltrametti and Maczyński (1994) and Länger and Maczyński (n.d.) proved a slightly modified version of the following result:

Theorem 3.4. The statements (3.1) and (3.2) are equivalent for any state p on a Boolean algebra L.

The formula in (3.1) defines a *correlation function of order n*, S_f^p, (with respect to a state p and a function f: $2^N \to \mathbb{R}$), i.e., S_f^p: $L^n \to \mathbb{R}$ such that

$$S_f^p(a_1, \ldots, a_n) = \sum_{I \subseteq N} f(I) p\left(\bigwedge_{i \in I} a_i\right) \qquad \text{for any} \quad a_1, \ldots, a_n \in L$$

In other words, S_f^p is a linear combination of $p(\wedge_{i \in I} a_i)$, and $p(\wedge_{i \in I} a_i)$ can be represented as a joint probability of a_i ($i \in I$) in the state p. The formal expression

$$0 \leq \sum_{I \subseteq N} f(I) p\left(\bigwedge_{i \in I} a_i\right) \leq 1 \tag{3.3}$$

is said to be a *Bell-type inequality of order n*. In view of Proposition 3.1, throughout the paper and its continuation, we shall understand by a Bell-type inequality only such an expression of the form (3.3), when f satisfies (3.2). We say that the Bell-type inequality (3.3) holds in an OML L and for a state p if it is true for all $a_1, \ldots, a_n \in L$. The inequality (3.3) generalizes the original inequality of Bell (1964),

$$p(a) + p(b) - p(a \wedge b) \leq 1$$

when in (3.3) we put $N = \{1, 2\}, f(\emptyset) = 0, f(\{1\}) = f(\{2\}) = -f(\{1, 2\}) = 1$.

Theorem 3.4 provides a very simple method of verification of Bell-type inequalities (Beltrametti and Maczyński, 1994): We take a function f: $2^N \rightarrow \mathbb{R}$ and we investigate whether it satisfies (3.2). Because N has n elements, this means we examine 2^n inequalities: If all sums occurring in (3.2) lie in the interval [0, 1], then the Bell inequality (3.3) holds in any classical model; in the opposite case we reject the inequality as not valid in any Kolmogorov probability model, and, in addition, in view of Propositions 3.1 and 3.2, also not valid in any OML. In the first case, this Bell-type inequality can be used as a test for a given system of events in order to see if it comes from a classical or nonclassical physical system.

In physical praxis, for f we take a function with $f(I) = 0$ whenever $\wedge_{i \in I} a_i$ is not physically measurable. The verification procedure simplifies when we consider a function f with integer values, in particular, when f: $2^N \rightarrow \{-1, 0, 1\}$. In general, for large values of n, some computational problems can appear (Pitowsky, 1989), but as we shall see, the most important cases are $n = 2$ and $n = 3$.

In view of Proposition 3.1, we see that it can happen that the conformation of the inequality (3.1) does not necessarily imply the classicality of the corresponding physical system; as remarked in Pulmannová and Majerník (1992), there are nonclassical systems with Bell-type inequalities (3.3) satisfying (3.2).

In addition, if, for example, a and b are not compatible, so that $a \wedge b$ is not commensurable, the values $p(a \wedge b)$ can also have a probabilistic interpretation in the analysis of nonclassical systems, i.e. (for more details see the end of Section 8)

$$p(a \wedge b) + p(a \wedge b^{\perp}) + p(a^{\perp} \wedge b) + p(a^{\perp} \wedge b^{\perp}) = 1$$

Therefore in what follows we shall concentrate on finer questions concerning the validity of Bell-type inequalities of type (3.3) in nonclassical models and with their interpretation, and we give a more detailed characterization of OMLs satisfying special types of Bell-type inequalities.

4. BELL-TYPE INEQUALITIES OF ORDER 2

The basic Bell (1964) inequality can be described in OMLs as follows:

$$p(a) + p(b) - p(a \wedge b) \leq 1, \qquad a, b \in L \tag{4.1}$$

This type, which is a particular form of (3.3), has been intensively studied in, e.g., Santos (1986, 1988), Pitowsky (1989), Beltrametti and Maczyński (1992a,b, 1994), and Pulmannová and Majerník (1992).

In the present section, we show that (4.1) holds in any OML for any subadditive or 0-additive state. We give some equivalent criteria for the validity of the Bell-type inequality (4.1) in OMLs.

Santos introduced the *separation* (Santos, 1986) or the *distance* (Santos, 1988) $S_p(a, b)$ between two propositions a, b in the state p via

$$S_p(a, b) = p(a) + p(b) - 2p(a \wedge b), \qquad a, b \in L \tag{4.2}$$

He proved that if L is a Boolean algebra, then S_p is a pseudometric on L.[5]

For any two elements a and b in L, we define

$$a\Delta b := (a \vee b) \wedge (a \wedge b)^{\perp}$$

and if p is a state on L we put

$$\rho_p(a, b) := p(a\Delta b), \qquad a, b \in L \tag{4.3}$$

Theorem 4.1. Let p be a state on an OML L. Then the following statements are equivalent:

(i) $p(a) + p(b) - p(a \wedge b) \leq 1$ for all $a, b \in L$ (Bell inequality).
(ii) p is subadditive.
(iii) p is a valuation.
(iv) p is 0-additive.
(v) ρ_p is a pseudometric on L.
(vi) S_p is a pseudometric on L.
(vii) $S_p = \rho_p$.
(viii) For every f: $2^{\{1,2\}} \to \mathbb{R}$ (f: $2^{\{1,2\}} \to \mathbb{Z}$) with

[5] A mapping S: $L \times L \to [0, \infty)$ is said to be a *pseudometric* on L if, for all $a, b, c \in L$, (i) $S(a, a) = 0$, (ii) $S(a, b) = S(b, a)$, (iii) $S(a, b) \leq S(a, c) + S(c, b)$ (*triangle inequality*).

$$\sum_{I\subseteq\{1,2\}} f(I)p\left(\bigwedge_{i\in I} a_i\right) \in [0, 1] \qquad \text{for any} \quad a_1, a_2 \in \{0, 1\}$$

we have

$$\sum_{I\subseteq\{1,2\}} f(I)p\left(\bigwedge_{i\in I} a_i\right) \in [0, 1] \qquad \text{for any} \quad a_1, a_2 \in L$$

(ix) For every $f\colon 2^{\{1,2\}} \to \mathbb{R}$ ($f\colon 2^{\{1,2\}} \to \mathbb{Z}$) with

$$\sum_{I\subseteq K} f(I) \in [0, 1] \qquad \text{for any} \quad K \subseteq \{1, 2\}$$

we have

$$\sum_{I\subseteq\{1,2\}} f(I)p\left(\bigwedge_{i\in I} a_i\right) \in [0, 1] \qquad \text{for any} \quad a_1, a_2 \in L$$

(x) There are a modular OML M, a homomorphism h from L onto M, and a (positive) subadditive state P on M such that $P(h(a)) = p(a)$ for any $a \in L$.
(xi) There exists an $\alpha > 0$ such that $1 - \alpha + \alpha p(a) + \alpha p(b) - \alpha p(a \wedge b) \leq 1$ for all $a, b \in L$.
(xii) There exists an $\alpha < 0$ such that $0 \leq -\alpha + \alpha p(a) + \alpha p(b) - \alpha p(a \wedge b)$ for all $a, b \in L$.
(xiii) $0 \leq p(b) - p(c) - p(a \wedge b) - p(b \wedge c) - p(c \wedge d) + p(a \wedge d)$ for all $a, b, c, d \in L$.[6]

Proof. The equivalence of (i)–(iii), (vi), (vii), and (xiii) has been proved in Pulmannová and Majerník (1992). Now let $a, b, c \in L$.

(iii) $\Rightarrow$ (iv). This is evident.

(iv) $\Rightarrow$ (iii). Since $a \leftrightarrow (a \wedge b)^\perp \leftrightarrow b$, due to the Foulis–Holland theorem[7] (Kalmbach, 1983; Pták and Pulmannová, 1991), $(a \vee b) \wedge (a \wedge b)^\perp = a \wedge (a \wedge b)^\perp \vee b \wedge (a \wedge b)^\perp$. On the other hand, $[a \wedge (a \wedge b)^\perp] \wedge [b \wedge (a \wedge b)^\perp] = 0$, which implies

$$\begin{aligned} &p(a \vee b) - p(a \wedge b) \\ &= p((a \vee b) \wedge (a \wedge b)^\perp) \end{aligned}$$

[6] This inequality is a Clauser–Horne-type inequality.
[7] *Foulis–Holland theorem*: If, for three elements $x, y, z \in L$, we have $x \leftrightarrow y \leftrightarrow z$, then the sublattice of L generated by $\{x, y, z\}$ is distributive.

$$= p(a \wedge (a \wedge b)^{\perp}) + p(b \wedge (a \wedge b)^{\perp})$$
$$= p(a) - p(a \wedge b) + p(b) - p(a \wedge b)$$

Hence, $p(a \vee b) + p(a \wedge b) = p(a) + p(b)$.

(ii) $\Rightarrow$ (v). According to Sarymsakov *et al.* (1983), the following inequality holds:

$$a\Delta b \leq (a\Delta c) \vee (b\Delta c) \tag{4.4}$$

Using the subadditivity of p and (4.4), we obtain (v).

(v) $\Rightarrow$ (iii). From the triangle inequality for ρ_p we conclude

$$\rho_p(a, b) \leq \rho_p(a, a \vee b) + \rho_p(a \vee b, b)$$
$$p(a \vee b) - p(a \wedge b) \leq p(a \vee b) - p(a) + p(a \vee b) - p(b)$$
$$p(a) + p(b) \leq p(a \vee b) + p(a \wedge b)$$

The last inequality also holds when we change a and b to $a^{\perp}$ and $b^{\perp}$, respectively, so that

$$p(a^{\perp}) + p(b^{\perp}) \leq p(a^{\perp} \vee b^{\perp}) + p(a^{\perp} \wedge b^{\perp})$$
$$p(a) + p(b) \geq p(a \vee b) + p(a \wedge b)$$

which implies that p is a valuation.

(viii) $\Leftrightarrow$ (ix). This equivalence follows from Proposition 3.2.

(i) $\Rightarrow$ (ix). Let f: $2^{\{1,2\}} \to \mathbb{R}$ (f: $2^{\{1,2\}} \to \mathbb{Z}$) satisfy (3.2) for $n = 2$. Define $A = f(\emptyset)$, $B = f(\{1\})$, $C = f(\{2\})$, and $D = f(\{1, 2\})$. Then (3.2) implies the following four inequalities:

$$0 \leq A \leq 1$$
$$0 \leq A + B \leq 1$$
$$0 \leq A + C \leq 1$$
$$0 \leq A + B + C + D \leq 1$$

Multiplying the above inequalities successively by the nonnegative numbers $1 - p(a) - p(b) + p(a \wedge b)$, $p(a) - p(a \wedge b)$, $p(b) - p(a \wedge b)$, and $p(a \wedge b)$, and summing all terms, we obtain

$$0 \leq A + Bp(a) + Cp(b) + Dp(a \wedge b) \leq 1$$

(ix) $\Rightarrow$ (i). This follows easily using the function f with $f(\emptyset) = 0, f(\{1\}) = f(\{2\}) = -f(\{1, 2\}) = 1$.

(ii) $\Rightarrow$ (x). It is not hard to show that

$$p((a \vee c)\Delta(b \vee c)) + p((a \wedge c)\Delta(b \wedge c)) \leq p(a\Delta b) \tag{4.5}$$

We write $a \sim b$ if $\rho_p(a, b) = 0$. From (v) and (4.5) we conclude that $\sim$ is a congruence on L, i.e., an equivalence relation on L such that if $a_i \sim b_i$, $i = 1, 2$, then $a_1 \vee a_2 \sim b_1 \vee b_2$, $a_1 \wedge a_2 \sim b_1 \wedge b_2$, and $a_i^\perp \sim b_i^\perp$, $i = 1, 2$. For any $a \in L$ put $[a] := \{b \in L: b \sim a\}$ and let $M := L/\sim := \{[a]: a \in L\}$. Then M is an OML with the least and greatest elements [0] and [1], the orthocomplementation $[a]^\perp := [a^\perp]$, the join $[a] \vee [b] = [a \vee b]$, and the meet $[a] \wedge [b] = [a \wedge b]$.

If $a \leq b$, then $[a] = [b]$ iff $p(a) = p(b)$. Now assume $[a] \leq [c]$. Put $c_1 := a \vee c$. Then $[c] = [c_1]$ and $a \leq c_1$. Since p is modular and $a \vee (b \wedge c_1) \leq (a \vee b) \wedge c_1$, we have

$$\begin{aligned}([a] \vee [b]) \wedge [c]) &= ([a] \vee [b]) \wedge [c_1] = [(a \vee b) \wedge c_1] \\ &= [a \vee (b \wedge c_1)] = [a] \vee ([b] \wedge [c_1]) = [a] \vee ([b] \wedge [c])\end{aligned}$$

which proves modularity of M.

The canonical mapping h: $L \to M$, defined via $a \mapsto [a]$, $a \in L$, is a homomorphism from L onto M.

We define a mapping P: $M \to [0, 1]$ by $P([a]) = p(a)$, $a \in L$. Since $\rho_p(a, b) = 0$ implies $0 \leq p(a \wedge (a \wedge b)^\perp)$,

$$p(b \wedge (a \wedge b)^\perp) \leq p((a \vee b) \wedge (a \wedge b)^\perp) = 0$$

and hence

$$\begin{aligned}p(a) &= p(a \wedge (a \wedge b)^\perp) + p(a \wedge b) \\ &= p(a \wedge b) = p(a \wedge b) \\ &= p(b \wedge (a \wedge b)^\perp) + p(a \wedge b) = p(b)\end{aligned}$$

we see that P is well defined.

If $[a] \perp [b]$, then $p(a) = p(a \wedge b^\perp)$ and $p(b) = p(a^\perp \wedge b)$ and hence

$$\begin{aligned}p(a \vee b) &\geq p((a \wedge b^\perp) \vee (a^\perp \wedge b)) \\ &= p(a \wedge b^\perp) + p(a^\perp \wedge b) \\ &= p(a) + p(b) \\ &\geq p(a \vee b)\end{aligned}$$

which shows that $P([a] \vee [b]) = P([a]) + P([b])$. Therefore, P is a subadditive state on M. In addition, $P([a]) = 0$ iff $p(a) = 0$, which is equivalent to $[a] = [0]$.

(x) $\Rightarrow$ (ii). This is evident.

(xi) $\Leftrightarrow$ (i). $1 - \alpha + \alpha p(a) + \alpha p(b) - \alpha p(a \wedge b) \leq 1$ iff $\alpha(p(a) + p(b) - p(a \wedge b)) \leq \alpha$.

(xii) $\Leftrightarrow$ (i). $0 \leq -\alpha + \alpha p(a) + \alpha p(b) - \alpha p(a \wedge b)$ iff $\alpha(p(a) + p(b) - p(a \wedge b)) \geq \alpha$. ∎

In Pulmannová and Majerník (1992) and Pták and Pulmannová (1994) it is shown that if the Bell inequality (4.1) holds for any p from an ordering or unital system $\mathcal{P}$ of states on L, then L is a Boolean algebra.

Now we present an example of an OML not a Boolean algebra in which the Bell inequality (4.1) holds for any state on it.

Example 4.2. There exists a non-Boolean OML L with the nonempty system of all states such that any Bell-type inequality of order 2 satisfying (3.2) holds for any state on L.

Proof. Let L be the OML from Proposition 2.4.10 in Pták and Pulmannová (1991), i.e., $L = L_1 \times L_2$, where L_1 is a stateless OML, $L_2 = \{0, 1\}$, and $L_1 \times L_2$ is the product OML of L_1 and L_2.[8]

Then L is a non-Boolean OML and on L there exists a unique state p, namely $p(a, 0) = 0$ and $p(a, 1) = 1$ for any $a \in L_1$. An easy calculation shows that p is subadditive. ■

It is well known that p defined via (2.4) is a subadditive state on $L(H)$ for a finite-dimensional Hilbert space H. This OML is not a Boolean algebra whenever dim $H > 1$; we recall that it is an important model for so-called finite-dimensional quantum mechanics (Busch *et al.*, 1993). In view of the equivalence of (ii) and (viii) in Theorem 4.1, we see that if p is *a priori* a subadditive state, then there is no Bell-type inequality, equivalently, no test using only pairs of propositions a and b which can decide whether the system under testing is classical or nonclassical.

Or it follows that the confirmation of any Bell-type inequality with $N = \{1, 2\}$ does not imply the classicality of the system.

In the following section, we shall deal in a more detailed way with subadditive states on $L(H)$.

Proposition 4.3. Let p be a state on an OML L, and let f: $2^{\{1,2\}} \to \mathbb{R}$ be a nonzero function. For the statements

(i) p is subadditive;
(ii) the mapping S_f^p: $L^2 \to \mathbb{R}$, defined by

$$S_f^p(a, b) := f(\emptyset) + f(\{1\})p(a) + f(\{2\})p(b) + f(\{1, 2\})p(a \wedge b), \qquad a, b \in L$$

is a pseudometric on L;
(iii) $f(\emptyset) = 0$ and $f(\{1\}) = f(\{2\}) = -f(\{1, 2\})/2 > 0$;

we have (ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i).

[8] Let $(L_i, \leq_i, 0_i, 1_i, {}^{\perp_i})_{i\in I}$ be a system of OMLs. Then $L = \Pi_{i\in I}L_i$ is an OML, where $\{a_i\}_i \leq \{b_i\}_i$ iff $a_i \leq_i b_i$ for any $i \in I$, and $\{a_i\}_i^{\perp} := \{a_i^{\perp_i}\}_i$ with the least and greatest elements $0 = \{0_i\}_i$ and $1 = \{1_i\}_i$, respectively. L is said to be the *product* OML of the system of OMLs $\{L_i\}_{i\in I}$.

Proof. (ii) $\Rightarrow$ (iii). This follows from $0 = S_f^p(1, 1) = S_f^p(0, 0) = f(\emptyset)$, and from $S_f^p(1, 0) = S_f^p(0, 1) \geq 0$.

(ii) $\Rightarrow$ (i). Let $a, b \in L$ and put $\alpha := f(\{1\})$. Then $S_f^p(a, b) = \alpha p(a) + \alpha p(b) - 2\alpha p(a \wedge b)$. In view of (ii), $S_f^p(a, b) \leq S_f^p(a, a \vee b) + S_f^p(a \vee b, b)$, which gives

$$\begin{aligned} &\alpha p(a) + \alpha p(b) - 2\alpha p(a \wedge b) \\ &\quad \leq \alpha p(a) + \alpha p(a \vee b) - 2\alpha p(a) + \alpha p(a \vee b) + \alpha p(b) - 2\alpha p(b) \end{aligned}$$

$$2\alpha p(a) + 2\alpha p(b) \leq 2\alpha p(a \vee b) + 2\alpha p(a \wedge b) \tag{4.6}$$

Changing a and b in (4.6) to $a^\perp$ and $b^\perp$, respectively, we finally obtain $p(a) + p(b) = p(a \vee b) + p(a \wedge b)$ for all $a, b \in L$, which is equivalent to the subadditivity of p. ■

5. BELL-TYPE INEQUALITIES OF ORDER 2 IN HILBERT SPACES

Let p be a state on an OML L. We say that an element $a_0 \in L$ is a *support* of p if $p(b) = 0$ for some $b \in L$ iff $b \perp a_0$. It is easy to show that if a support of p exists, it is unique:

$$a_0 = \bigwedge\{a \in L: p(a) = 1\}$$

and $p(a_0) = 1$.

Proposition 5.1. If a_0 is the support of a subadditive state on an OML L, then $a_0 \in C(L)$.

Proof. Let a be an arbitrary element of L. Using the valuation property of p, we have

$$\begin{aligned} 1 &\geq p(a_0 \wedge a \vee a_0 \wedge a^\perp) \\ &= p(a_0 \wedge a) + p(a_0 \wedge a^\perp) \\ &= p(a_0) + p(a) - p(a_0 \vee a) + p(a_0) + p(a^\perp) - p(a_0 \vee a^\perp) \\ &= 3 - p(a_0 \vee a) - p(a_0 \vee a^\perp) \geq 1 \end{aligned}$$

Therefore, $p(a_0 \wedge a \vee a_0 \wedge a^\perp) = 1$. Since $a_0 \wedge a \vee a_0 \wedge a^\perp \leq a_0$ and since a_0 is the support of p, we conclude that $a_0 \wedge a \vee a_0 \wedge a^\perp = a_0$. Since a was an arbitrary element of L, this implies $a_0 \in C(L)$. ■

It is well known that if H is a finite-dimensional Hilbert space, $1 \leq \dim H = n$, then p defined via (2.4) is a subadditive state on $L(H)$. We show that any subadditive state on $L(H)$, $1 \leq \dim H < \infty$, has the form (2.4). It is worth saying that we do not have to use the Gleason theorem, which holds only for $L(H)$ with dim $H \geq 3$. We recall that if x is a nonzero vector in H, then M_x denotes the one-dimensional subspace of H generated by x.

Proposition 5.2. Let $1 \leq \dim H = n < \infty$. Then we have:

(i) Any subadditive state on $L(H)$ has a support.
(ii) The support of a subadditive state on $L(H)$ is equal to H.
(iii) Any subadditive state p on $L(H)$ is of the form (2.4).

Proof. (i) Let p be a subadditive state on $L(H)$ and put $L_p^1 := \{M \in L(H): p(M) = 1\}$. Then $L_p^1 \neq \emptyset$. Let $M_0 := \wedge\{M: M \in L_p^1\}$. It is clear that $M_0 \in L(H)$. Since the dimension of H is finite, there are finitely many elements $M_1, \ldots, M_k$ of L_p^1 such that $M_0 = \wedge_{i=1}^k M_i$. Then $p(M_0^\perp) = p(\vee_{i=1}^k M_i^\perp) \leq \Sigma_{i=1}^k p(M_i^\perp) = 0$, which proves that M_0 is the support of p.

(ii) It is well known that for any H, $L(H)$ is irreducible.[9] Since for the support M_0 of p we have $p(M_0) = 1$, Proposition 5.1 implies $M_0 = H$.

(iiia) If dim $H = 1$, then the assertion (iii) is trivial. So assume dim $H = 2$. We claim that, for any unit vector $x \in H$, $p(M_x) = 1/2$. If not, we can find such an M_x such that $p(M_x) < 1/2$ (or we choose $M_x^\perp$). It is easy to find a unit vector $y \in H\backslash(M_x \cup M_x^\perp)$ such that $p(M_y) \leq 1/2$. Then $p(M_x \vee M_y) = p(H) = 1 > p(M_x) + p(M_y)$, which contradicts the subadditivity of p. Hence, p is of the form (2.4).

(iiib) Let $3 \leq \dim H = n < \infty$. We assert that, for all unit vectors x and y in H, we have $p(M_x) = p(M_y)$. Indeed, if x and y are linearly dependent, then $M_x = M_y$, so that $P(M_x) = p(M_y)$. If x and y are linearly independent, they generate a two-dimensional subspace H_0 of H. The mapping p_0 on $L(H_0)$, defined via

$$p_0(M) := p(M)/p(H_0), \qquad M \in L(H_0)$$

is a subadditive state on $L(H_0)$ [we note that $p(H_0) > 0$ in view of (ii)]. According to part (iiia), $p_0(M_x) = p_0(M_y)$, which gives $p(M_x) = p(M_y)$.

Choose an orthonormal basis $\{x_i\}_{i=1}^n$ in H; then $1 = p(H) = \Sigma_{i=1}^n p(M_{x_i}) = np(M_x)$, which gives $p(M_x) = 1/n$ for any unit vector $x \in H$; consequently, p has the form (2.4). ■

Proposition 5.3. Let dim $H = \infty$. Then we have:

(i) If p is a subadditive state on $L(H)$, then $p(M) = 0$ for any finite-dimensional subspace M of H.
(ii) There is no subadditive, completely additive state on $L(H)$.

Proof. (i) Let p be a subadditive state on $L(H)$. Assume $p(M_0) > 0$ for some finite-dimensional subspace M_0 of H. For two given linearly independent

[9] For example, for $M \in L(H)$, $\{0\} \neq M \neq H$, choose a unit vector $x \in H \setminus (M \cup M^\perp)$. Then $M_x \geq M_x \wedge M \vee M_x \wedge M^\perp = \{0\}$.

unit vectors x and y, define $H_0 = M_0 \vee \mathrm{sp}(x, y)$.[10] Applying Proposition 5.2 to the subadditive state p_0 on $L(H_0)$, defined via $p_0(M) := p(M)/p(H_0)$, $M \in L(H_0)$, we conclude that $p(M_x) = p(M_y)$. Put $\lambda := p(M_x)$, and choose a countable, infinite orthonormal system $\{x_i\}_{i=1}^{\infty}$ in H. Let $H_n = \mathrm{sp}(x_1, \ldots, x_n)$. Then

$$p(H) \geq p(H_n) = n\lambda$$

for any $n \geq 1$. Hence, $\lambda = 0$; consequently, $p(M) = 0$ for any finite-dimensional subspace M of H.

(ii) Assume there exists some a subadditive, completely additive state p on $L(H)$. Choose an ONB $\{x_i\}$ in H. In view of (i), $1 = p(H) = p(\bigvee_i M_{x_i}) = \Sigma_i\, p(M_{x_i}) = 0$, which is a contradiction. ∎

We recall that in Sarymsakov *et al.* (1983, p. 62), using the method of uniformities on quantum logics, it has been proved that on $L(H)$, dim $H = \aleph_0$, there is no separating system of subadditive, completely additive states.

We note that due to Aarnes (1970; Dvurečenskij, 1993), any finitely additive measure on $L(H)$ can be uniquely expressed as the sum

$$p = p_1 + p_2 \tag{5.1}$$

where p_1 is a completely additive measure on $L(H)$ and p_2 is a finitely additive measure on $L(H)$ vanishing on any finite-dimensional subspace of H. In addition, if dim $H = \infty$, then by Alda (1980), there is no 0–1-valued state on $L(H)$.

For a generalization of Proposition 5.3, see Proposition 6.8.

The result of Proposition 5.2 can be generalized as follows. Let $S(H)$ denote the set of all skew operators on H, i.e., of all linear operators P: $H \to H$ such that $P^2 = P$. Then any idempotent operator is continuous (Dunford and Schwartz, 1957) and $S(H)$ contains as a subset the set $\mathcal{P}(H)$ of all orthogonal projections on H, i.e., of all Hermitian idempotents on H.

Put $E := \mathrm{Ran}P = \{Px\colon x \in H\} = \{x \in H\colon x = Px\} \in L(H)$ and $F := \mathrm{Ker}P = \{x \in H\colon Px = 0\} \in L(H)$. Then $E \cap F = \{0\}$ and $E + F = H$, and P projects any vector $x \in H$ onto E parallel with F. This relationship among P, E, and F will be written as $P = \pi(E, F)$. If, for $E, F \in L(H)$ we have $E \cap F = \{0\}$ and $E + F = H$, then E, F determine a unique skew operator $P = \pi(E, F) \in S(H)$. Indeed, we put $Px = x_1$, $x \in H$, whenever $x = x_1 + x_2$, $x_1 \in E$, $x_2 \in F$.

We have $I - \pi(E, F) = \pi(F, E)$ and $\pi^*(E, F) = \pi(F^\perp, E^\perp)$, where I is the identity operator on H.

[10] For every subset M of a vector space V, let sp$M$ denote the linear subspaces of V generated by M.

We endow $S(H)$ with internal structures: For $P, Q \in S(H)$, we write $P \leq Q$ iff $PQ = QP = P$, and $P^{\perp} := I - P$. Then $S(H)$ is an OMP[11,12] with respect to $\leq$ and $^{\perp}$, defined above, and, in addition, if $P \perp Q$, i.e., $P \leq Q^{\perp}$, or equivalently $PQ = QP = O$, where O is the null operator on H, then $P \vee Q = P + Q$. We recall that $\pi(E_1, F_1) \leq \pi(E_2, F_2)$ iff $E_1 \subseteq E_2$ and $F_2 \subseteq F_1$.

It is possible to show that if dim $H \leq 2$, then $S(H)$ is an OML. If dim $H \geq 3$, then $S(H)$ is not an OML. This follows from the following statement: If

$$\pi(E_1, F_1) \vee \pi(E_2, F_2) = \pi(E, F) \tag{5.2}$$

and $\pi(E, F) \neq I$, then $E_1 \vee E_2 = E$ and $F = F_1 \cap F_2$ [here $\vee$ is taken in $L(H)$]; conversely, if $E = E_1 \vee E_2$, $F = F_1 \cap F_2$, $E \cap F = \{0\}$, and $E + F = H$, then (5.2) holds. [For a more general statement see Mushtari (1989).]

Indeed, it is clear that $E_0 := E_1 \vee E_2 \subseteq E$ and $F_1 \cap F_2 \supseteq F$. If, for example, $E_0 \neq E$, there exists a unit vector x in E orthogonal to E_0. Choose a Hamel basis $\{x_i: i \in I\}$ of E containing the vector $x = x_{i_0}$ such that $\{x_i: i \in I\}\setminus\{x\}$ is a Hamel basis of $E \cap M_x^{\perp}$ and a Hamel basis $\{y_j: j \in J\}$ of F. Take a vector y_{j_0}, $j_0 \in J$, and put

$$\tilde{E} = \mathrm{sp}(\{x_i: i \in I\setminus\{i_0\}\} \cup \{x_{i_0} - y_{j_0}\})$$

Then $\tilde{E} \cap F = \{0\}$, $\tilde{E} + F = H$, and $\pi(\tilde{E}, F)$ dominates $\pi(E_1, F_1)$ and $\pi(E_2, F_2)$, but $\pi(\tilde{E}, F)$ and $\pi(E, F)$ are not comparable, which contradicts (5.2), and hence $E_0 = E$. Since $P^* \leq Q^*$ iff $P \leq Q$, changing π to π^* in (5.2), we obtain $F_1 \cap F_2 = F$.

It is worth saying that if $\pi(E_1, F_1) \perp \pi(E_2, F_2)$, then

$$\pi(E_1, F_1) \vee \pi(E_2, F_2) = \pi(E_1, F_1) + \pi(E_2, F_2) = \pi(E_1 + E_2, F_1 \cap F_2) \tag{5.3}$$

Indeed, we have $E_1 + E_2 \subseteq E$, $F_1 \cap F_2 \supseteq F$, where $\pi(E, F)$ is that from (5.2). If $x \in E$, then

$$x = \pi(E, F)x = \pi(E_1, F_1)x + \pi(E_2, F_2)x \in E_1 + E_2$$

Choose a skew operator $P \in S(H)$ and define

$$A = P^*P + (I - P)^*(I - P) \tag{5.4}$$

[11] An *orthomodular poset* (OMP) is a poset L with a partial ordering $\leq$, least and greatest elements 0 and 1, and orthocomplementation $^{\perp}: L \to L$ such that for all $a, b \in L$ we have (i) $a^{\perp\perp} = a$, (ii) if $a \leq b$, then $b^{\perp} \leq a^{\perp}$, (iii) if $a \leq b^{\perp}$, then $a \vee b \in L$, (iv) if $a \leq b$, then $b = a \vee a^{\perp}$.

[12] A nonnegative mapping p on an OMP L is said to be (i) a *state* if $p(1) = 1$, and, if $p(a \vee b) = p(a) + p(b)$ whenever $a \perp b$, (ii) *subadditive* if $p(a \vee b) \leq p(a) + p(b)$ whenever $a \vee b$ exists in L.

Then A is a Hermitian, positive invertible operator on H [$A^{-1} = P_{\mathrm{Ran}\, P}P^* + P_{\mathrm{Ker}\, P}(I - P^*)$] and it defines on H a new inner product $(\cdot, \cdot)_A$ via

$$(x, y)_A := (Ax, y), \qquad x, y \in H$$

Then H with respect to $(\cdot, \cdot)_A$ is again a Hilbert space, and the topologies induced by $\|\cdot\|$ and $\|\cdot\|_A$ are the same.

Let $\mathcal{P}_A(H)$ denote the set of all orthogonal projections on H with respect to $(\cdot, \cdot)_A$. We note that $\mathcal{P}(H) = \mathcal{P}_I(H)$. Then

$$S(H) = \bigcup_A \mathcal{P}_A(H)$$

where A is defined via (5.4), or $S(H) = \bigcup_A \mathcal{P}_A(H)$, where A is any positive invertible operator on H. In addition, $P \in \mathcal{P}_A(H) \Leftrightarrow AP = P^* A \Leftrightarrow P^* \in \mathcal{P}_{A^{-1}}(H)$.

Proposition 5.4. Let $1 \leq \dim H = n < \infty$. Then on $S(H)$ there is a unique state, which is subadditive on any $\mathcal{P}_A(H)$, namely p: $S(H) \to [0, 1]$, defined via

$$p(\pi(E, F)) := \dim E/n, \qquad P = \pi(E, F) \in S(H) \tag{5.5}$$

Proof. Define p on $S(H)$ via (5.5). Since $P_1 = \pi(E_1, F_1) \perp P_2 = \pi(E_2, F_2)$ iff $E_1 \subseteq F_2$ and $E_2 \subseteq F_1$, we have $E_1 \cap E_2 \subseteq E_1 \cap F_1 = \{0\}$. According to (5.3), we have

$$p(P_1 \vee P_2) = \dim(E_1 \vee E_2)/n = \dim E_1/n + \dim E_2/n = p(P_1) + p(P_2)$$

Finally, $p(I) = p(\pi(H, \{0\})) = 1$, which proves that (5.5) defines a state on $S(H)$.

We assert that if $P_1, P_2 \in \mathcal{P}_A(H)$, then $P_1 \vee_A P_2$ exists in $\mathcal{P}_A(H)$ [$\vee_A$ denotes the join taken in $\mathcal{B}_A(H)$] as well as $P_1 \vee P_2$ exists in $S(H)$, and both are equal. Indeed, let $P_i = \pi(E_i, F_i)$, $i = 1, 2$. Then $P_1 \vee_A P_2 = \pi(E_1 \vee_A E_2, F_1 \cap F_2)$. But

$$E_1 \vee_A E_2 = (\mathrm{sp}(E_1 \cup E_2))^{-A} = (\mathrm{sp}(E_1 \cup E_2))^{-} = E_1 \vee E_2$$

where ${}^{-A}$ and ${}^{-}$ denote the closure with respect to $\|\cdot\|_A$ and $\|\cdot\|$, respectively. On the other hand, $F_1 \wedge_A F_2 = F_1 \cap F_2 = F_1 \wedge F_2$. Therefore, p defined via (5.5) is subadditive on any $\mathcal{P}_A(H)$.

Now assume that p is a state on $S(H)$ which is subadditive on any $\mathcal{P}_A(H)$. If P is an orthogonal projection onto a closed subspace $M \in L(H)$, then $P = \pi(M, M^\perp) = P_M \in S(H)$. The restriction of p onto the set $\mathcal{P}(H)$ of all orthogonal projections on H is a subadditive state on $\mathcal{P}(H)$ [we note that if $P_M, P_N \in \mathcal{P}(H)$, then the join of P_M and P_N in $S(H)$ exists and is equal to $P_{M \vee N}$]. By Proposition 5.2 [we can identify $M \in L(H)$ with $P_M \in \mathcal{P}(H)$]

$$p(\pi(M, M^{\perp})) = \dim M/n, \qquad M \in L(H)$$

Therefore the restriction of p to $\mathcal{P}_A(H)$ is a subadditive state on $\mathcal{P}_A(H)$, so that p is the state in question. ∎

It is worth saying that the former proposition has been proved without referring to Gleason's theorem; the extension of Gleason's theorem to $S(H)$ can be found in Mushtari (1989). We recall that (5.5) can be rewritten as

$$p(P) = \operatorname{tr}(P)/n, \qquad P \in S(H)$$

where tr denotes the trace. Indeed, let $P = \pi(E, F) \in S(H)$ and choose an ONB $\{x_i\}_i$ in E and an ONB $\{y_j\}_j$ in $E^{\perp}$. Then

$$\begin{aligned} \operatorname{tr}(P)/n &= \sum_i (Px_i, x_i) + \sum_j (Py_j, y_j) \\ &= \dim E/n + \sum_j (y_j, P^*y_j) \\ &= \dim E/n + \sum_j (y_j, \pi(F^{\perp}, E^{\perp})y_j) \\ &= \dim E/n \end{aligned}$$

In addition, according to Mushtari and Matvejchuk (1985), we can show that any state on $S(H)$, $2 \neq \dim H < \infty$, has the form (5.5). We may obtain the same result for $S(H)$ when H is an n-dimensional complex Hilbert space, $n \geq 3$, without applying Gleason's theorem, from Mushtari and Matvejchuk (1985): Let f be a positive bounded linear functional on the set $B(H)$ of all bounded operators on H, which is an extension of the state p on $S(H)$ (Yeadon, 1983, 1984; Christensen, 1982). Due to Petz and Zemánek (1988, Theorem 2), the condition $\sup\{f(P): P \in S(H)\} < \infty$ is equivalent to f being tracial on $B(H)$. By Kadison and Ringrose (1986, Example 8.1.2), on $B(H)$ there is a unique tracial functional f with $f(I) = 1$, namely $f(A) = c\operatorname{tr}(A)$, $A \in B(H)$; consequently $p(P) = \operatorname{tr}(P)/n$, $P \in S(H)$.

Remark 5.5. (i) If $\dim H = 2$, then on $S(H)$ there is a unique subadditive state, namely p given by (5.5).

(ii) If $\dim H \geq 3$, there is no subadditive state on $S(H)$.

Proof. (i) Checking all possibilities in $S(H)$, we can see that p defined by (5.5) is the unique subadditive state on $S(H)$.

(ii) Take $E_1, E_2 \in L(H)$ such that $\dim E_1 = n - 2$, $\dim E_2 = 1$, $E_2 \underset{(\neq)}{\subset} E_1$, and let F_1, F_2 be complements of E_1 and E_2, respectively, such that

$F_1 \cap F_2 \subseteq E_1 \vee E_2$. Then $\pi(E_1, F_1) \vee \pi(E_2, F_2) = I$, but $\dim E_1 + \dim E_2 < n$, which proves that p defined by (5.5) is a state on $S(H)$, which is not subadditive on $S(H)$ [it is subadditive only on every $\mathcal{P}_A(H)$]. ∎

Theorem 5.6. If p is a Jauch–Piron state on an OML L and $p^{-1}(\{1\})$ satisfies the d.c.c.,[13] then p has a support.

Proof. Suppose p has no support. Since 1 is not the support of p, there exists an $a_1 \in L\backslash\{1\}$ with $p(a_1) = 1$. Since a_1 is not the support of p, there exists an element $a_2 \in L$ with $a_1 \not\leq a_2$ and $p(a_2) = 1$. Because of $a_1 \not\leq a_2$, we have $a_1 \wedge a_2 < a_1$. The Jauch–Piron property of p implies $p(a_1 \wedge a_2) = 1$. Since $a_1 \wedge a_2$ is not the support of p, there exists an $a_3 \in L$ with $a_1 \wedge a_2 \leq a_3$ and $p(a_3) = 1$. Because of $a_1 \wedge a_2 \not\leq a_3$, we have $a_1 \wedge a_2 \wedge a_3 < a_1 \wedge a_2$. The Jauch–Piron property of p implies $p(a_1 \wedge a_2 \wedge a_3) = 1$. Going on in this way, one obtains an infinite strictly descending chain $1 > a_1 > a_1 \wedge a_2 > a_1 \wedge a_2 \wedge a_3 > \cdots$ in $p^{-1}(\{1\})$, contradicting our assumption. Therefore, p has a support. ∎

Remark 5.7. Every $L(H)$, if $\dim H < \infty$, satisfies the d.c.c.; consequently, every Jauch–Piron state on $L(H)$, $\dim H < \infty$, has a support; compare Proposition 5.2.

6. OTHER SUBADDITIVE STATES

In this section, we present subadditive states on other OMLs. First we show that if H is a two-dimensional Hilbert space, then the assertion (iii) of Proposition 5.2 is a particular case of the following result concerning subadditive states on horizontal sums of Boolean algebras.

Let $(L_i, \leq_i, {}^{\perp_i}, 0_i, 1_i)_{i \in I}$ be a nonvoid system of OMLs such that (i) $0 := 0_i$, $1 =: 1_i$ for any $i \in I$, and (ii) $L_i \cap L_j = \{0, 1\}$ for any $i, j \in I$, $i \neq j$. By $L = \Sigma_{i \in I} L_i$ we denote the *horizontal sum* of $(L_i)_{i \in I}$, where $L = \bigcup_{i \in I} L_i$. We put $a \leq b$ if there exists some $i \in I$ such that $a, b \in L_i$ and $a \leq_i b$. For all $i \in I$ and all $a \in L_i$, we put $a^\perp := a^{\perp_i}$. An easy calculation shows that L together with $\leq$, ${}^\perp$, 0, 1 is a well-defined OML.

Proposition 6.1. Let $L = \Sigma_{i \in I} B_i$ be the horizontal sum of a system of Boolean algebras $\{B_i\}_{i \in I}$, where $|I| > 1$ and $|B_i| > 2$ for any $i \in I$. Then there is a subadditive state on L, say p, if and only if $|B_i| = 4$ for any $i \in I$. In this case, p is unique and $p(a) = 1/2$ for any $a \in L\backslash\{0, 1\}$.

[13] We say that a poset P with a partial ordering $\leq$ satisfies the *descending chain condition* (d.c.c.) if there do not exist infinitely many elements $a_1, a_2, \ldots$ of P with $a_1 > a_2 > \ldots$.

Proof. If $|B_i| = 4$ for any $i \in I$, then it is easy to see that $p: L \to [0, 1]$ such that $p(0) = 0, p(1) = 1$, and $p(a) = 1/2$, $a \in L\backslash\{0, 1\}$, is a subadditive state on L.

Conversely, let p be a subadditive state on $L = \Sigma_{i \in I} B_i$. Suppose there is an $a \in L\backslash\{0, 1\}$ such that $p(a) < 1/2$. Choose any $b \in L\backslash\{0, 1, a\}$ with $p(b) \leq 1/2$. Then $p(a \vee b) = p(1) = 1 > p(a) + p(b)$, which contradicts the subadditivity of p.

Since, for arbitrarily fixed $i \in I$, $p|B_i$ is a state on B_i and $p(a) = 1/2$ for any $a \in B_i\backslash\{0, 1\}$, from the equality $a = a \wedge b \vee a \wedge b^{\perp}$, which holds for any $b \in B_i$, in particular for any $a, b \in B_i\backslash\{0, 1, a\}$, we conclude that $|B_i| = 4$. ∎

We can obtain another interesting class of projections on a Hilbert space considering *Krein spaces*. So let H be a Hilbert space and P any orthogonal projection on H. We put $J = P - (I - P)$ and define a new inner product $[\cdot, \cdot]$ on $H \times H$ via

$$[x, y] := (Jx, y), \qquad x, y \in H$$

Then $[\cdot, \cdot]$ is not necessarily a positive inner product. Denote by $\Gamma^+ := \{x \in H: [x, x] = 1\}$, $\Gamma^- := \{x \in H: [x, x] = -1\}$, $\Gamma^0 := \{x \in H: [x, x] = 0\}$, and $H^+ := PH$, $H^- := (I - P)H$. A vector $x \in H$ is said to be *isotropic* if $x \in \Gamma^0$.

For example, if $H = \mathbb{R}^2$ and $\dim H^+ = 1$, then $\Gamma^+ \cup \Gamma^-$ consists of two hyperbolas $x^2 - y^2 = \pm 1$, and Γ^0 consists of two lines $y = \pm x$. If $H = \mathbb{R}^3$ and $\dim H^+ = 2$, then two rotational hyperboloids $x^2 + y^2 - z^2 = \pm 1$ form $\Gamma^+ \cup \Gamma^-$. We recall that the case $H = \mathbb{R}^4$ and $\dim H^+ = 1$ is used as a Minkowski space in special relativity theory. For more information on Krein spaces see, e.g., Azizov and Yokhvidov (1986) and on their applications in physics see Nagy (1966).

Denote by $K_J(H)$ the set of all idempotent linear operators P on H such that $[Px, y] = [x, Py]$ for all $x, y \in H$, or equivalently $JP = P^*J$. For two elements $P, Q \in K_J(H)$ we write $P \leq Q$ iff $PQ = QP = P$, and put $P^{\perp_J} := I - P$. Then $K_J(H)$ with respect to $\leq$ and $\perp_J$ is an OMP with the least and greatest elements O and I, respectively. We recall that $P \perp_J Q$ iff $PQ = QP = O$, and then $P \vee Q$ exists in $K_J(H)$ and $P \vee Q = P + Q$. If $P = I$, then $J = I$ and $K_J(H) = \mathcal{P}(H)$.

For any subspaces M of H we put $M^{\perp_J} := \{x \in H: [x, y] = 0 \text{ for all } y \in M\}$. A subspace M of H is said to be *projectively complete* if $M + M^{\perp_J} = H$. It is possible to show that $P \in K_J(H)$ iff $P = \pi(M, M^{\perp_J})$ for some projectively complete subspace M; in abbreviation, we write $P = \pi(M)$.

We note that M is a one-dimensional projectively complete subspace of H iff $P = \pi(\mathrm{sp}(x)) = [x, x][\cdot, x]x$ for some $x \in \Gamma^+ \cup \Gamma^-$. In addition, for any subspace M of H we have $M^{\perp_J} = JM^{\perp}$, and it is a mirror image of M with respect to the axis H^+.

It is possible to show, similarly as for $S(H)$, that if

$$\pi(M_1) \vee \pi(M_2) = \pi(M) \tag{6.1}$$

then $M_1 \vee M_2 \subseteq M$, where $\vee$ is taken in $L(H)$, and if $\pi(M_1) \perp_J \pi(M_2)$, then $M = M_1 \vee M_2 = M_1 + M_2$. Unfortunately, in general, $M_1 \vee M_2 \neq M$, as shown in Example 6.4.

Example 6.2. Assume dim $H = 2$. Then $K_J(H)$ is an OML, and any subadditive state p on $K_J(H)$ has the form

$$p(\pi(M)) = \dim M/2, \qquad \pi(M) \in K_J(H)$$

Proof. This follows from the fact that $K_J(H)$ is a horizontal sum of four-element Boolean algebras and from Proposition 6.1. ■

An element $P \in K_J(H)$ is said to be *positive* (*negative*) if $[Px, x] > 0$ ($[Px, x] < 0$) for any nonzero $x \in PH$. It is well known (Azizov and Yokhvidov, 1986, Theorem 6.4) that every element $P \in K_J(H)$ can be expressed as

$$P = P^+ + P^-$$

where P^+ and P^- are positive and negative elements of $K_J(H)$, respectively. Equivalently, if $P = \pi(M)$, then

$$M = M^+ + M^-$$

where $P^+ = \pi(M^+)$ and $P^- = \pi(M^-)$. According to the inertia law (Azizov and Yokhvidov, 1986, Theorem 6.5), if $P = P_1 + P_2$, where P_1 and P_2 are positive and negative elements of $K_J(H)$, respectively, then

$$\dim M^+ = \dim M_1 \qquad \text{and} \qquad \dim M^- = \dim M_2$$

whenever $M = M_1 + M_2$ and M_1 and M_2 are positive and negative subspaces of H.

Proposition 6.3. Let $2 \leq \dim H = n < \infty$ and $\kappa := \min\{\dim H^+, \dim H^-\} \geq 1$. Then any of the functions

$$p_1(\pi(M)) = \dim M/n, \qquad \pi(M) \in K_J(H) \tag{6.2}$$

$$p_2(\pi(M)) = \dim M^+/\dim H^+, \qquad \pi(M) \in K_J(H) \tag{6.3}$$

$$p_3(\pi(M)) = \dim M^-/\dim H^-, \qquad \pi(M) \in K_J(H) \tag{6.4}$$

defines a state on $K_J(H)$. Consequently, any convex linear combination $p = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3$, $\lambda_1, \lambda_2 \lambda_3 \geq 0$, $\lambda_1 + \lambda_2 + \lambda_3 = 1$, of p_1, p_2, p_3 is a state on $K_J(H)$, too.

Proof. If $P = P^+ + P^- \in K_J(H)$ and $Q = Q^+ + Q^- \in K_J(H)$, where P^+, Q^+ and P^-, Q^- are positive and negative, then

$$P \vee Q = P + Q = (P + Q)^+ + (P + Q)^- = (P^+ + Q^+) + (P^- + Q^-)$$

and $P^+ + Q^+$ and $P^- + Q^-$ are positive and negative. Using the inertia law, we can prove the assertions of the proposition. ■

We recall that (6.2) can be rewritten equivalently as

$$p_1(P) = \mathrm{tr}(P)/n, \qquad P \in K_J(H)$$

Example 6.4. (i) If $H = \mathbb{R}^3$, $\dim H^+ = 2$, then $K_J(H)$ is a nonmodular OML and p_1, p_2, p_3 are states on $K_J(H)$ which are not subadditive.

(ii) If $\kappa = 2$, then $K_J(H)$ is not a lattice.

Proof. (i) In this case, the inner product $[\cdot, \cdot]$ has the form $[(x, y, z), (x, y, z)] = x^2 + y^2 - z^2$, $(x, y, z) \in H$. Isotropic vectors lie on cones determined by $x^2 + y^2 = z^2$. If $\dim M_1 \geq 1$, $\dim M_2 = 2$, then by (6.1) we have either $\pi(M_1) \vee \pi(M_2) = \pi(M_2)$ or $\pi(M_1) \vee \pi(M_2) = \pi(H)$. If $\dim M_1 = \dim M_2 = 1$, $M_1 \neq M_2$, then $M_1 + M_2$ is either projectively complete, and then $\pi(M_1) \vee \pi(M_2) = \pi(M_1 + M_2)$, or $M_1 + M_2$ is not projectively complete, and then $\pi(M_1) \vee \pi(M_2) = \pi(H)$.

Take a nonzero isotropic vector $x \in H$ and let M_x be the subspace of H generated by x. Then M_x is not projectively complete because $M_x \cap M_x^{\perp_J} = M_x$. In $M_x^{\perp_J}$ we can find two independent nonisotropic vectors y and z. Then M_y and M_z are projectively complete and, in view of $\mathrm{sp}(y, z) = M_x^{\perp_J}$, $\pi(M_y) \vee \pi(M_z) = \pi(H)$, which proves that p_1 in (6.2) is not subadditive, consequently $K_J(H)$ is not modular.

Varying y and z such that both are negative and positive, we see that p_2 in (6.3) and p_3 in (6.4) are not subadditive. We recall that p_3 is a 0–1-valued state.

(ii) Choose two linearly independent nonzero isotropic vectors x and y. In $\mathrm{sp}(x, y)$ we can find two linearly independent nonisotropic vectors u and v. Then M_u and M_v are projectively complete, but $\pi(M_u) \vee \pi(M_v)$ does not exist in $K_J(H)$.

For more information on the extension of Gleason's theorem to $K_J(H)$ see Matvejchuk (1989, 1991).

The assertions of Proposition 5.2 can be generalized. Before that we present the following result [for a σ-additive variant see Pták and Pulmannová (1991, Theorem 2.3.2)].

Lemma 6.5. Let $L = L_1 \times \cdots \times L_n$ be the product of OMLs $L_1, \ldots, L_n$. Then we have:

(i) A real-valued mapping p on L is a state on L if and only if there exist a nonempty subset $I \subseteq \{1, \ldots, n\}$, positive numbers $\alpha_i > 0$, $i \in I$, with $\Sigma_{i \in I} \alpha_i = 1$, and states p_i on L_i for every $i \in I$ such that

$$p((a_1, \ldots, a_n)) = \sum_{i \in I} \alpha_i p_i(a_i), \qquad (a_1, \ldots, a_n) \in L \tag{6.5}$$

In addition, this representation is unique.

(ii) On L there is no state if and only if, for every $i = 1, \ldots, n$, there is no state on L_i.

(iii) A state p on L is subadditive if and only if, for any $i \in I$, the corresponding state p_i on L_i in (6.5) is subadditive.

(iv) We have

$$|S(L)| = 0 \qquad \text{if} \quad \sum_{i=1}^{n} |S(L_i)| = 0$$

$$|S(L)| = 1 \qquad \text{if} \quad \sum_{i=1}^{n} |S(L_i)| = 1$$

$$|S(L)| = 2^{\aleph_0} \cdot \sum_{i=1}^{n} |S(L_i)| \qquad \text{if} \quad \sum_{i=1}^{n} |S(L_i)| > 1$$

Proof. (i) Let p be a state on L. Define $I := \{i \in \{1, \ldots, n\}: p(1^i) > 0\}$, where $1^i := (a_1, \ldots, a_n)$ with $a_j = 0_j$ if $j \neq i$ and $a_j = 1_i$ if $j = i$. Then, for any $i \in I$, the mapping $p_i: L_i \to [0, 1]$, defined by $p_i(a) := p(a^i)/p(1^i)$ for all $a \in L_i$, where $a^i = (a_1, \ldots, a_n)$ with $a_j = 0_j$ for $j \neq i$ and $a_j = a$ for $j = i$, is a state on L_i. Now (6.5) holds, where $\alpha_i = p(1^i)$, $i \in I$. The uniqueness of the representation (6.5) is evident.

It is clear that (6.5) with given properties on the α_i's, $I \subseteq \{1, \ldots, n\}$, and p_i, $i \in I$, defines a state on L.

(ii) This follows from (i).

(iii) If each p_i, $i \in I$, is subadditive, by (6.5) p is subadditive, too. Conversely, let p be subadditive. Then, for any $i \in I$ and all $a, b \in L_i$, we have

$$\alpha_i p_i(a \vee b) = p(a^i \vee b^i) \leq p(a^i) + p(b^i) = \alpha_i(p_i(a) + p_i(b))$$

which proves the subadditivity of p_i.

(iv) Put $s := \Sigma_{i=1}^{n} |S(L_i)|$. Then $|S(L)| = 0, 1, 2^{\aleph_0}$, or $\max(|S(L_1)|, \ldots, |S(L_n)|)$, depending on whether $s = 0$, $s = 1$, $1 < s \leq n$, or $s \geq 2^{\aleph_0}$, respectively. ∎

We say that an OML L has finite *rank* if there is an integer k such that any set of mutually orthogonal nonzero elements in L has at most k elements. The least such integer k is said to be the rank of L.

Proposition 6.6. Let L be an irreducible modular OML of rank n. Then on L there is a unique subadditive state p, namely

$$p(a) = \dim a/n, \qquad a \in L \tag{6.6}$$

where $\dim a$ is the cardinality of a maximal set of mutually orthogonal atoms less than or equal to a. In addition, 1 is the support of p.

Proof. According to Varadarajan (1968, Theorem 2.8), the number $\dim a$ does not depend on the chosen maximal set of mutually orthogonal atoms $\leq a$, and (6.6) defines a subadditive state on L.

Conversely, assume that p is a subadditive state on L. If p does not have the form (6.6), then there is an $a \in L$ with $p(a) \neq (\dim a)/n$. Clearly, $0 < \dim a < n$. Let B and C be maximal sets of mutually orthogonal atoms less than or equal to a and $a^\perp$, respectively. Then there exist two atoms $u, v \in B \cup C$ such that $p(u) < p(v)$. Moreover, there exists an atom x of L with $x \leq u \vee v$, $u \neq x \neq v$, and $p(x) \leq p(u \vee v)/2$. Such a possibility of finding a third atom x less than $u \vee v$ follows from Varadarajan (1968, Theorem 2.15). Then $p(u \vee x) = p(u \vee v) = p(u) + p(v) > p(u) + p(x)$, contradicting the subadditivity of p. Hence, p has the form (6.6).

The fact that 1 is the support of p is now evident. ■

We note that if H is an n-dimensional Hilbert space, then $L(H)$ satisfies the conditions of Proposition 6.6; consequently, any subadditive state on $L(H)$ has the form (6.6), or equivalently (2.4) (see Proposition 5.2).

Remark 6.7. Let L be a modular OML of finite rank, and $C(L)$ be its center. Since $C(L)$ is of finite rank, too, there are finitely many elements $c_1, \ldots, c_k \in C(L)$ such that (i) $c_i \wedge c_j = 0$ for $i \neq j$, $c_1 \vee \cdots \vee c_k = 1$; (ii) $C(L)$ is precisely the set of all atoms of the form $c_{i_1} \vee \cdots \vee c_{i_r}$ $(1 \leq i_1, \ldots, i_r \leq k)$. For every $i = 1, \ldots, k$, $L_i := L_{[0,c_i]} = \{b \in L: b \leq c_i\}$ is an OML with respect to $\perp_i$ defined by $b^{\perp_i} = b^\perp \wedge c_i$ for all $b \in L_i$. Then L is isomorphic to $L_1 \times \cdots \times L_k$ [$a \mapsto (a \wedge c_1, \ldots, a \wedge c_k)$ is a corresponding isomorphism] and any L_i is an irreducible modular OML of finite rank (Varadarajan, 1968, Theorem 2.14). Hence, for any subadditive state p on L, there exist by Lemma 6.5 and Proposition 6.6 a nonempty set $I \subseteq \{1, \ldots, k\}$, and positive numbers $\alpha_i > 0$, $i \in I$, with $\sum_{i \in I} \alpha_i = 1$, such that

$$p(a) = \sum_{i \in I} \alpha_i \dim(a \wedge c_i)/\dim c_i, \qquad a \in L \tag{6.7}$$

Conversely, any mapping p defined by (6.7) is a subadditive state on L. Indeed, if $a, b \in L$, using the Foulis–Holland theorem, we have

$$
\begin{aligned}
p(a \vee b) &= \sum_{i \in I} \alpha_i \dim((a \vee b) \wedge c_i)/\dim c_i \\
&= \sum_{i \in I} \alpha_i \dim((a \wedge c_i) \vee (b \wedge c_i))/\dim c_i \\
&\leq \sum_{i \in I} \alpha_i \dim(a \wedge c_i)/\dim c_i + \sum_{i \in I} \alpha_i \dim(b \wedge c_i)/\dim c_i \\
&= p(a) + p(b)
\end{aligned}
$$

In addition, the support of this subadditive state p on L is equal to $a_0 := \vee_{i \in I} c_i$. Indeed, due to (6.7), for $a \in L$, $p(a) = 1$ iff $\dim(a \wedge c_i) = \dim c_i$ for any $i \in I$, or, equivalently, $a \wedge c_i = c_i$ for any $i \in I$, hence iff $a \geq a_0$.

Proposition 6.8. Let L be an irreducible, complete, atomic[14] OML satisfying the exchange axiom,[15] in which there exists an infinite set of mutually orthogonal atoms.

(i) If p is a subadditive state on L, then $p(a) = 0$ for any $a \in L$ with $\dim a < \infty$.

(ii) There is no subadditive, completely additive state on L.

Proof. By Kalmbach (1986, Theorems 8.20, 8.17), there exists a map dim: $L \to [0, \infty]$ such that $\dim 0 = 0$, $\dim a < \infty$ iff $L_{[0,a]}$ is modular, and $\dim a$ is equal to the maximal number of mutually orthogonal atoms less than or equal to a.

(i) Let p be a subadditive state on L. We claim $p(a) = 0$ whenever $\dim a < \infty$. Indeed, suppose there exists an $e \in L$. We claim $p(a) = 0$ and $p(e) > 0$. Choose an $f \in L$ with $e \leq f$ and $(\dim e)/p(e) < \dim f < \infty$. Since all atoms in $L_{[0,f]}$ have dimension equal to 0, $L_{[0,f]}$ is an irreducible (Kalmbach, 1986, Theorem 10.8), modular OML of finite rank. Applying Proposition 6.6 to the subadditive state $p/p(f)$ on $L_{[0,f]}$, one obtains $p(e)/p(f) = (\dim e)/(\dim f)$. But $(\dim e)/(\dim f) < p(e)$, which is absurd, hence $p(a) = 0$ whenever $\dim a < \infty$.

(ii) Suppose that p is a subadditive, completely additive state on L. Let $\{a_i\}$ be a maximal system of mutually orthogonal atoms in L. Then $1 = \vee_i a_i$. Hence, by (i), $1 = p(1) = p(\vee_i a_i) = \Sigma_i\, p(a_i) = 0$, which gives a contradiction. ■

[14] An element $a \neq 0$ of an OML L is said to be an *atom* of L if $b \leq a$ for $b \in L$ implies $b \in \{0, a\}$. An OML L is said to be *atomic* if, for any $b \in L \backslash \{0\}$, there is an atom a of L with $a \leq b$.

[15] We say that an element y of an OML L *covers* $x \in L$ if $x \leq y$, $x \neq y$, and if $x \leq z \leq y$, $z \in L$, imply $z \in \{x, y\}$. The OML L satisfies the *exchange axiom* if, for $a, b \in L$, we have: If a covers $a \wedge b$, then $a \vee b$ covers b.

7. BOOLEAN ALGEBRAS AND BELL-TYPE INEQUALITIES OF ORDER 2

In the present section, we give some conditions characterizing Boolean algebras among OMLs via sets of subadditive states, which in view of Theorem 4.1 correspond to states satisfying the Bell inequality (4.1).

We recall that M_4 denotes a lattice with the base set $\{e, a, b, c, d, f\}$ such that $e \leq a, b, c, d \leq f$ and a, b, c, d are mutually different and not comparable, i.e., its Hasse diagram is given by Fig. 2.

Theorem 7.1. If an OML L is not a Boolean algebra, then it contains a sublattice isomorphic to M_4 containing 0 and a sublattice isomorphic to M_4 containing 1.

Proof. Let $a, b \in L$ with $a \not\leftrightarrow b$ be given. Put $c := \text{com}(a, b) < 1$ [(2.3)] and define $a_1 := a \wedge c^\perp$, $a_2 := a^\perp \wedge c^\perp$, $a_3 := b \wedge c^\perp$, and $a_4 := b^\perp \wedge c^\perp$. Then we have $a_1 = a \wedge (a^\perp \vee b) \wedge (a^\perp$:VU $b^\perp)$, $a_2 = a^\perp$:VD $(a \vee b) \wedge (a \vee b^\perp)$, $a_3 = b \wedge (a \vee b^\perp) \wedge (a^\perp \vee b^\perp)$ and $a_4 = b^\perp \wedge (a \vee b) \vee (a^\perp \vee b)$. Since $a \not\leftrightarrow b$, we have $a^\perp < (a^\perp \vee b) \wedge (a^\perp \vee b^\perp)$ and hence $a_1 \neq 0$. Similarly we have $a_i \neq 0$ for all i.

Obviously, $a_1 \wedge a_2 = 0$. Moreover,

$$a_1 \wedge a_3 = (a \wedge c^\perp) \wedge (b \wedge c^\perp) = (a \wedge b) \wedge c^\perp$$
$$= (a \wedge b) \wedge (a^\perp \vee b^\perp) = 0$$

In a similar way it follows that $a_i \wedge a_j = 0$ for $i \neq j$. Now $a_1 \vee a_2 = a \wedge c^\perp \vee a^\perp \wedge c^\perp = (a \vee a^\perp) \wedge c^\perp = c^\perp$ and $a_1 \vee a_3 = a \wedge c^\perp \vee b \wedge c^\perp = (a \vee b) \wedge c^\perp = c^\perp$. In a completely analogous way one obtains $a_i \vee a_j = c^\perp$ for $i \neq j$.

Since $a_i \wedge a_j = 0$ for $i \neq j$ and $a_i \neq 0$ for all i, the a_i are pairwise distinct. The existence of an integer $i \in \{1, \ldots, 4\}$ with $a_i = c^\perp$ would

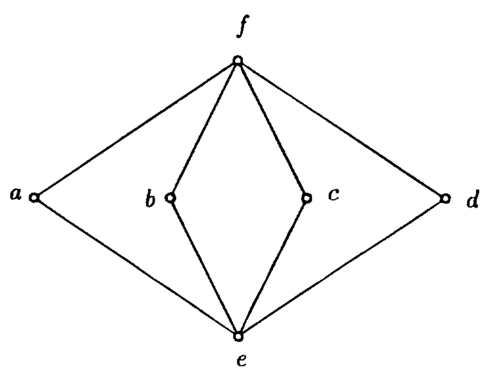

Fig. 2.

imply $a_j = a_j \wedge c^{\perp} = a_j \wedge a_i = 0$ for $j \neq i$, which is a contradiction, too. Therefore, $c^{\perp} \neq a_i$ for all i. This shows that $\{0, a_1, \ldots, a_4, c^{\perp}\}$ is a sublattice of L isomorphic to M_4 and containing 0. Clearly $\{c, a_1^{\perp}, \ldots, a_4^{\perp}, 1\}$ is a sublattice of L isomorphic to M_4 and containing 1. ■

Lemma 7.2. Let L be an OML which is not a Boolean algebra and p a subadditive state on L. Then there exist $A, B \subseteq L\backslash\{0, 1\}$ with $|A| = |B| = 4$ such that $p(a) = p(b) \leq 1/2$ for all $a, b \in A$ and $p(c) = p(d) \geq 1/2$ for all $c, d \in B$.[16]

Proof. By Theorem 7.1, there exists a sublattice L_1 of L isomorphic to M_4 and containing 0. Let A denote the four-element antichain of L_1. Then $A \subseteq L\backslash\{0, 1\}$ and, for all $a, b \in A$, we have $p(a) = p(a \vee c) - p(c) = p(b \vee c) = p(b)$, where $c \in A\backslash\{a, b\}$. Moreover, $p(a) = [p(a) + p(b)]/2 = p(a \vee b)/2 \leq 1/2$ for $a, b \in A$, $a \neq b$.

The second statement follows dually. ■

Theorem 7.3. Let L be an OML which has a set $\mathcal{P}$ of subadditive states such that, for every $A \subseteq L\backslash\{0, 1\}$ with $|A| = 4$, there exists a $p \in \mathcal{P}$ with $|p(A)| > 1$. Then L is a Boolean algebra.

Proof. If L were not a Boolean algebra, then, by Lemma 7.2, there would exist an $A \subseteq L\backslash\{0, 1\}$ with $|A| = 4$ such that $|p(A)| = 1$ for all $p \in \mathcal{P}$, contradicting our assumption. Hence L is a Boolean algebra. ■

Theorem 7.4. Let L be an OML which has a set $\mathcal{P}$ of subadditive states such that, for every $A \subseteq L\backslash\{0, 1\}$ with $|A| = 4$, there exist a $p \in \mathcal{P}$ and an $a \in A$ with $p(a) > 1/2$. Then L is a Boolean algebra.

Proof. If L were not a Boolean algebra, then, by Lemma 7.2, there would exist an $A \subseteq L\backslash\{0, 1\}$ with $|A| = 4$, such that $p(a) \leq 1/2$ for all $p \in \mathcal{P}$ and all $a \in A$, contradicting our assumption. Hence, L is a Boolean algebra. ■

Theorem 7.5. Let L bean OML which has a set $\mathcal{P}$ of subadditive states such that, for every $A \subseteq L\backslash\{0, 1\}$ with $|A| = 4$, there exists a $p \in \mathcal{P}$ and an $a \in A$ with $p(a) < 1/2$. Then L is a Boolean algebra.

Proof. This is analogous to the previous proof. ■

Theorem 7.6. Let L be an OML which has a generating set M and a set $\mathcal{P}$ of subadditive states with the property that $a \in L$, $b \in M$, and $p(a) \leq p(b)$ for all $p \in \mathcal{P}$ imply $a \leq b$. Then L is Boolean algebra.

[16]One of the authors is grateful to Dr. M. Navara for calling his attention to these facts.

Proof. Let $a, b \in M$. Then, for all $p \in \mathcal{P}$, we have

$$\begin{aligned} p(a \wedge (a^\perp \vee b)) &= p(a) + p(a^\perp \vee b) - 1 \\ &= -p(a^\perp) + p(a^\perp) + p(b) - p(a^\perp \wedge b) \\ &\leq p(b) \end{aligned}$$

whence $a \wedge (a^\perp \vee b) \leq b$. Since $a \wedge (a^\perp \vee b) \leq a$, we conclude $a \wedge (a^\perp \vee b) \leq a \wedge b$, and therefore $a \wedge (a^\perp \vee b) = a \wedge b$. But this is equivalent to $a \leftrightarrow b$.

Remarks 7.7. (i) Theorem 7.1 follows also from the fact that an OML is a Boolean algebra iff it neither contains MO2 nor MO2 $\times$ 2^1 as a subalgebra. (This follows from the following results (Beran, 1984, Chapter III.2): (a) The free OML F_2 of rank 2 is isomorphic to MO2 $\times$ 2^4. (b) The subalgebra generated by two noncommuting elements of an OML must be a nondistributive homomorphic image of F_2, i.e., it must be isomorphic to MO2 $\times$ 2^i for some $i \in \{0, \ldots, 4\}$. (c) MO2 $\times$ 2^i is isomorphic to a subalgebra of MO2 $\times$ 2^j for $1 \leq i \leq j$.

(ii) If an OML L has an ordering (Pulmannová and Majerník, 1992, Theorem 4) or full system of subadditive states, then the conditions of theorem 7.3 are fulfilled.

(iii) Theorem 7.4 generalizes the known result that an OML having a unital set of subadditive states (Pták and Pulmannová, 1994) is a Boolean algebra.

(iv) Theorem 7.6 generalizes the assertion (Pulmannová and Majerník, 1992, Theorem 4) that an OML having an ordering set of subadditive states has to be a Boolean algebra.

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